THE LEAST COMMON MULTIPLE OF SEQUENCE OF PRODUCT OF LINEAR POLYNOMIALS

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ABSTRACT. Let $f(x)$ be the product of several linear polynomials with integer coefficients. In this paper, we obtain the estimate: $\log \mathrm{lcm}(f(1), ..., f(n)) \sim An$ as $n \to \infty$, where A is a constant depending on f.

1. Introduction

The first significant attempt for proving the prime number theorem was made by Chebyshev in 1848-1852. In fact, Chebyshev [2] introduced the following two functions:

$$\vartheta(x) := \sum_{p\le x} \log p = \log \prod_{p\le x} p \quad \text{and} \quad \psi(x) := \sum_{p^k\le x} \log p = \log \mathrm{lcm}_{1\le i\le \lfloor x\rfloor}\{i\},$$

where p denotes a prime number and $\lfloor x\rfloor$ denotes the greatest integer no more than x. The prime number theorem asserts that $\vartheta(n) \sim \psi(n) \sim n$. Thus the asymptotic formula $\log \mathrm{lcm}_{1\le i\le n}\{i\} \sim n$ is equivalent to the prime number theorem. From then on, the topic of estimating the least common multiple of any given sequence of positive integers become prevalent and important. Hanson [6] and Nair [9] got the upper bound and lower bound of $\mathrm{lcm}_{1\le i\le n}\{i\}$ respectively. Farhi [3] investigated the least common multiple of arithmetic progression while Farhi and Kane [4] and Hong and Yang [8] studied the least common multiple of consecutive positive integers. Recently, Hong and Qian [7] got some results on the least common multiple of consecutive arithmetic progression terms. In 2002, Bateman, Kalb and Stenger [1] proved that for any integers a and b such that $a \ge 1$ and $a + b \ge 1$ and $\gcd(a, b) = 1$, one has

$$\log \mathrm{lcm}_{1\le i\le n}\{ai + b\} \sim \frac{an}{\varphi(a)} \sum_{\substack{r=1\\ \gcd(r,a)=1}}^{a} \frac{1}{r}$$

as $n \to \infty$, where $\varphi(a)$ denotes the number of integers relatively prime to a between 1 and a.

Let h and l be any two relatively prime positive integers. The renowned Dirichlet's theorem says that there are infinitely many prime numbers in the arithmetic progression $\{hm + l\}_{m\in\mathbb{N}}$. Furthermore, if we define

$$\vartheta(x; h, l) := \sum_{\substack{\text{prime } p\le x\\ p\equiv l \pmod h}} \log p,$$

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then the prime number theorem for arithmetic progressions says that

$$\vartheta(x;h,l)=\frac{x}{\varphi(h)}+o(x), \tag{1.1}$$

For an analytic proof, see Davenport [5]. Moreover, Selberg [10] gave an elementary proof of this result.

In this paper, we concentrate on the least common multiple of sequence of product of linear polynomials with integer coefficients. Throughout this paper, for any polynomial $g(x)=a_nx^n+\ldots+a_0$ with integer coefficients, we define

$$L_n(g):=\operatorname{lcm}(g(1),\ldots,g(n)).$$

If $\gcd(a_0,\ldots,a_n)=d$, then

$$\log L_n(g)=\log(dL_n(g_1))=\log L_n(g_1)+\log d=\log L_n(g_1)+O(1),$$

where $g_1(x)=\frac{a_n}{d}x_n+\ldots+\frac{a_0}{d}$ is a primitive polynomial. Thus it suffices to give the estimate for primitive polynomials. As usual, let $\mathbb{Q}$ and $\mathbb{N}$ denote the field of rational numbers and the set of nonnegative integers. Define $\mathbb{N}^*:=\mathbb{N}\backslash\{0\}$. For any prime number p, we let v_p be the normalized p-adic valuation of $\mathbb{N}^*$, i.e., $v_p(a)=s$ if $p^s\parallel a$. For any two positive integers a and b, let $\langle b\rangle_a$ denote the least nonnegative integer congruent to b modulo a between 0 and $a-1$. We can now state the main result of this paper.

Theorem 1.1. *Let $\{t_i\}_{i=0}^k$ be an increasing sequence of integers with $t_0=0$, and let $\{d_j\}_{j=1}^{t_k}$ be a sequence of positive integers such that $d_1=\cdots=d_{t_1}>d_{t_1+1}=\cdots=d_{t_2}>\cdots>d_{t_{k-1}+1}=\cdots=d_{t_k}$. Let*

$$f(x):=\prod_{i=1}^{k}\prod_{j=t_{i-1}+1}^{t_i}(a_jx+b_j)^{d_j},$$

where $a_j,b_j\in\mathbb{N}^*$ and $\gcd(a_j,b_j)=1$ for each $1\le j\le t_k$ and $a_{j_1}b_{j_2}\ne a_{j_2}b_{j_1}$ for any two integers $1\le j_1\ne j_2\le t_k$. Then we have

$$\log L_n(f)\sim\frac{n}{\varphi(q)}\sum_{\substack{r=1\\ \gcd(r,q)=1}}^{q}\sum_{i=1}^{k}(d_{t_i}-d_{t_{i+1}})\max_{1\le j\le t_i}\{\frac{a_j}{\langle b_jr\rangle_{a_j}}\} \tag{1.2}$$

as $n\to\infty$, where $q=\operatorname{lcm}_{1\le j\le t_k}\{a_j\}$ and $d_{t_{k+1}}:=0$.

Note that if some b_j in Theorem 1.1 are negative integers, then (1.2) is still true. Evidently, if one picks $k=t_1=d_{t_1}=1$, then Theorem 1.1 reduces to the Bateman-Kalb-Stenger theorem [1]. The proof of Theorem 1.1 will be given in the second section.

2. **Proof of Theorem 1.1**

In this section, we prove the main theorem. For convenience, in what follows we let $g_j(x):=a_jx+b_j$ for all $1\le j\le t_k$. Then $f(x)=\prod_{i=1}^k\prod_{j=t_{i-1}+1}^{t_i}g_j(x)^{d_j}$. We can now show Theorem 1.1 as follows.

Proof of Theorem 1.1. Let $q=\operatorname{lcm}_{1\le j\le t_k}\{a_j\}$, and let $R(q)=\{r\in\mathbb{N}^*\mid 1\le r\le q,\gcd(r,q)=1\}$ be the set of positive integers relatively prime to q and not exceeding q. In the following, we let $P_n(f)$ be the set of the prime factors of $L_n(f)$, and let $P_{n,1}(f)$ denote the set of the elements in $P_n(f)$ which divide either $\operatorname{lcm}_{1\le j_1\ne j_2\le t_k}\{(a_{j_1}b_{j_2}-$

$a_{j_2}b_{j_1})\}$ or q. Define $P_{n,2}(f) := P_n(f)\backslash P_{n,1}(f)$ to be the complementary set of $P_{n,1}(f)$ in $P_n(f)$. Obviously we have

$$L_n(f) = \Big(\prod_{p\in P_{n,1}(f)} p^{v_p(L_n(f))}\Big)\Big(\prod_{p\in P_{n,2}(f)} p^{v_p(L_n(f))}\Big),$$

equivalently,

$$\log L_n(f) = \sum_{p\in P_{n,1}(f)} v_p(L_n(f))\log p + \sum_{p\in P_{n,2}(f)} v_p(L_n(f))\log p. \tag{2.1}$$

We claim that if $p\in P_{n,2}(f)$ and $p|f(m)$ for some positive integer m, then there is a unique integer j_0 with $1\le j_0\le t_k$ such that $p|g_{j_0}(m)$ and $p\nmid g_j(m)$ for all other integers j between 1 and t_k. In fact, we suppose that $p|(a_{j_1}m+b_{j_1})$ and $p|(a_{j_2}m+b_{j_2})$ for some positive integers j_1 and j_2 with $1\le j_1\ne j_2\le t_k$. Then we have

$$p|a_{j_1}(a_{j_2}m+b_{j_2}) - a_{j_2}(a_{j_1}m+b_{j_1}) = a_{j_1}b_{j_2} - a_{j_2}b_{j_1}.$$

It follows that

$$p|\mathrm{lcm}_{1\le j\ne l\le t_k}\{a_jb_l - a_lb_j\},$$

which means that $p\in P_{n,1}(f)$. This is a contradiction. So the claim is proved. Thus for any $p\in P_{n,2}(f)$, by the claim we have

$$\begin{aligned} v_p(L_n(f)) &= \max_{1\le m\le n}\{v_p(f(m))\} = \max_{1\le m\le n}\Big(\sum_{j=1}^{t_k} d_jv_p(g_j(m))\Big) = \max_{1\le m\le n}\max_{1\le j\le t_k}\{d_jv_p(g_j(m))\} \\ &= \max_{1\le j\le t_k}\max_{1\le m\le n}\{d_jv_p(g_j(m))\} = \max_{1\le j\le t_k}\{d_jv_p(L_n(g_j))\}. \end{aligned} \tag{2.2}$$

If $p\in P_{n,2}(f)$ and $\max_{1\le j\le t_k}\{v_p(L_n(g_j))\}\ge 2$, then we have $p^2|L_n(g_{j_0})$ for some integer $j_0\in[1,t_k]$, which implies that $p^2|g_{j_0}(m)$ for some positive integers $m\le n$. Therefore

$$p\le \sqrt{g_{j_0}(m)} \le \sqrt{g_{j_0}(n)} \le M_n := \max_{1\le j\le t_k}\{\sqrt{g_j(n)}\}. \tag{2.3}$$

On the other hand, by the definition of $P_{n,1}(f)$, we obtain that $P_{n,1}(f)$ consists of only finitely many primes, and hence for all primes $p\in P_{n,1}(f)$ and all sufficiently large n, we have $p\le M_n \ll \sqrt{n}$. Thus for all sufficiently large n, we can rewrite (2.1) as

$$\log L_n(f) = \sum_{p\le M_n} v_p(L_n(f))\log p + \sum_{\substack{p>M_n\\ p\in P_{n,2}(f)}} v_p(L_n(f))\log p. \tag{2.4}$$

It is obvious that if $p\le M_n$, then

$$v_p(L_n(f)) \le \frac{\log f(n)}{\log p} = \sum_{j=1}^{t_k} d_j\frac{\log g_j(n)}{\log p} \ll \sum_{j=1}^{t_k} d_j\frac{\log n}{\log p} \ll \frac{\log n}{\log p}.$$

Thus we have

$$\begin{aligned} \sum_{p\le M_n} v_p(L_n(f))\log p &\ll \sum_{p\le M_n}\frac{\log n}{\log p}\log p \ll \sum_{p\le M_n}\log n \ll \pi(M_n)\log n \\ &\ll \frac{M_n}{\log M_n}\log n \ll \frac{\sqrt{n}}{\log\sqrt{n}}\log n \ll \sqrt{n}. \end{aligned}$$

It then follows from (2.4) that

$$\log L_n(f) = \sum_{\substack{p>M_n\\ p\in P_{n,2}(f)}} v_p(L_n(f))\log p + O(\sqrt{n}). \tag{2.5}$$

Now let $p \in P_{n,2}(f)$. Then it is easy to see that p is congruent to r' modulo q for some $r' \in R(q)$ and $\gcd(r', a_j) = 1$ for all $1 \le j \le t_k$. For such r', there is exactly one $r \in R(q)$ such that $rr' \equiv 1 \pmod q$, and hence we have $\langle b_j r\rangle_{a_j} p \equiv \langle b_j r\rangle_{a_j} r' \equiv b_j rr' \equiv b_j \pmod{a_j}$ for all $1 \le j \le t_k$. We can easily derive that $\langle b_j r\rangle_{a_j} p$ is the smallest multiple of p which is congruent to b_j modulo a_j for all $1 \le j \le t_k$. It follows that for any $1 \le j \le t_k$ and any $p \in P_{n,2}(f)$ which is congruent to r' modulo q, we have that $p|(a_j m + b_j)$ for some $m \le n$ if and only if $p \le \frac{a_j n+b_j}{\langle b_j r\rangle_{a_j}}$. Thus, for any sufficiently large n and for any prime $p \in P_{n,2}(f)$ which is congruent to $r' \in R(q)$ modulo q satisfying $p > M_n$, we have by (2.2) and (2.3) that

$$v_p(L_n(f)) = \max_{1\le j\le t_k}\{d_j v_p(L_n(g_j))\} = \max_{1\le j\le t_k}\{e_j\},$$

where

$$e_j := \begin{cases} d_j, & \text{if } M_n < p \le \frac{a_j n+b_j}{\langle b_j r\rangle_{a_j}},\\ 0, & \text{if } p > \frac{a_j n+b_j}{\langle b_j r\rangle_{a_j}}. \end{cases}$$

Since $d_1 = \cdots = d_{t_1} > d_{t_1+1} = \cdots = d_{t_2} > \cdots > d_{t_{k-1}+1} = \cdots = d_{t_k}$, we deduce that for sufficiently large n,

$$v_p(L_n(f)) = \begin{cases} d_{t_1}, & \text{if } M_n < p \le \max_{1\le j\le t_1}\{\frac{a_j n+b_j}{\langle b_j r\rangle_{a_j}}\},\\ d_{t_i}, & \text{if } \max_{1\le j\le t_{i-1}}\{\frac{a_j n+b_j}{\langle b_j r\rangle_{a_j}}\} < p \le \max_{1\le j\le t_i}\{\frac{a_j n+b_j}{\langle b_j r\rangle_{a_j}}\} \text{ for some } 2\le i\le k. \end{cases}$$

Obviously, we have that for sufficiently large n, $v_p(L_n(f)) = 0$ for any prime $p > M_n$ and $p \notin P_{n,2}(f)$. Thus we obtain by (2.5) that

$$\begin{aligned}
\log L_n(f) &= \sum_{p>M_n} v_p(L_n(f))\log p + O(\sqrt{n})\\
&= \sum_{r'\in R(q)} \sum_{\substack{p>M_n,\\ p\equiv r' \pmod q}} v_p(L_n(f))\log p + O(\sqrt{n})\\
&= \sum_{r'\in R(q)} \Bigg(\sum_{\substack{M_n<p\le \max_{1\le j\le t_1}\{\frac{a_j n+b_j}{\langle b_j r\rangle_{a_j}}\}\\ p\equiv r' \pmod q}} d_{t_1}\log p\\
&\qquad + \sum_{i=2}^{k} \sum_{\substack{\max_{1\le j\le t_{i-1}}\{\frac{a_j n+b_j}{\langle b_j r\rangle_{a_j}}\}<p\le \max_{1\le j\le t_i}\{\frac{a_j n+b_j}{\langle b_j r\rangle_{a_j}}\}\\ p\equiv r' \pmod q}} d_{t_i}\log p\Bigg) + O(\sqrt{n})\\
&= \sum_{r'\in R(q)} \Bigg(d_{t_1}\Big(\vartheta\big(\max_{1\le j\le t_1}\{\frac{a_j n+b_j}{\langle b_j r\rangle_{a_j}}\}; q, r'\big) - \vartheta(M_n; q, r')\Big) + \sum_{i=2}^{k} d_{t_i} F_i(n)\Bigg) + O(\sqrt{n}),
\end{aligned}$$

where

$$F_i(n) := \vartheta\Big(\max_{1\le j\le t_i}\{\frac{a_j n+b_j}{\langle b_j r\rangle_{a_j}}\}; q, r'\Big) - \vartheta\Big(\max_{1\le j\le t_{i-1}}\{\frac{a_j n+b_j}{\langle b_j r\rangle_{a_j}}\}; q, r'\Big).$$

Now, applying the prime number theorem for arithmetic progressions (i.e. (1.1)), we obtain that

$$\log L_n(f)=\frac{n}{\varphi(q)}\sum_{r'\in R(q)}\sum_{i=1}^{k}d_{t_i}\Big(\max_{1\le j\le t_i}\{\frac{a_j}{\langle b_j r\rangle_{a_j}}\}-\max_{1\le j\le t_{i-1}}\{\frac{a_j}{\langle b_j r\rangle_{a_j}}\}\Big)+o(n),$$

where $\max_{1\le j\le t_0}\{\frac{a_j}{\langle b_j r\rangle_{a_j}}\}:=0$ and $rr'\equiv 1 \pmod q$.

Since r runs over $R(q)$ as r' does, it follows immediately that

$$\begin{aligned}\log L_n(f)&=\frac{n}{\varphi(q)}\sum_{r\in R(q)}\sum_{i=1}^{k}d_{t_i}\Big(\max_{1\le j\le t_i}\{\frac{a_j}{\langle b_j r\rangle_{a_j}}\}-\max_{1\le j\le t_{i-1}}\{\frac{a_j}{\langle b_j r\rangle_{a_j}}\}\Big)+o(n)\\&=\frac{n}{\varphi(q)}\sum_{r\in R(q)}\Big(d_{t_k}\max_{1\le j\le t_k}\{\frac{a_j}{\langle b_j r\rangle_{a_j}}\}+\sum_{i=1}^{k-1}(d_{t_i}-d_{t_{i+1}})\max_{1\le j\le t_i}\{\frac{a_j}{\langle b_j r\rangle_{a_j}}\}\Big)+o(n).\end{aligned}$$

So we get (1.2) and Theorem 1.1 is proved. □

In particular, we have the following two consequences.

Corollary 2.1. *Let $l\ge 1$ be an integer and $\{s_i\}_{i=1}^{l}$ be a decreasing sequence of positive integers, and let $g(x)=\prod_{i=1}^{l}(a_ix+b_i)^{s_i}$, where $a_i,b_i\in\mathbb{N}^*$ and $\gcd(a_i,b_i)=1$ for each $1\le i\le l$ and $a_ib_j\ne a_jb_i$ for any $1\le i\ne j\le l$. Then we have*

$$\log L_n(g)\sim\frac{n}{\varphi(q)}\sum_{\substack{r=1\\ \gcd(r,q)=1}}^{q}\sum_{i=1}^{l}(s_i-s_{i+1})\max_{1\le j\le i}\{\frac{a_j}{\langle b_j r\rangle_{a_j}}\},$$

where $q=\mathrm{lcm}_{1\le i\le l}\{a_i\}$ and $s_{l+1}:=0$.

Corollary 2.2. *Let $l,d\ge 1$ be integers and $g(x)=\prod_{i=1}^{l}(a_ix+b_i)^{d}$, where $a_i,b_i\in\mathbb{N}^*$ and $\gcd(a_i,b_i)=1$ for each $1\le i\le l$ and $a_ib_j\ne a_jb_i$ for any two integers $1\le i\ne j\le l$. Then we have*

$$\log L_n(g)\sim\frac{dn}{\varphi(q)}\sum_{\substack{r=1\\ \gcd(r,q)=1}}^{q}\max_{1\le i\le l}\{\frac{a_i}{\langle b_i r\rangle_{a_i}}\},$$

where $q=\mathrm{lcm}_{1\le i\le l}\{a_i\}$.

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